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# THE MARTINGALES OF AN INDEPENDENT INCREMENT PROCESS

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A simple proof is given of the representation of martingales adapted to the sigma fields of a process with stationary, independent increments.

Processes with stationary independent increments
representation of martingales
stochastic integrals

## 1. Introduction

In 1967, Kunita and Watanabe [3] proved that if $B_t$ is a Brownian motion, then any square integrable martingale adapted to the sigma fields of $B_t$ could be represented as a stochastic integral with respect to $B_t$, a result that has proved very useful in filtering and control theory.

If $X_t$ is a process with stationary, independent increments, finding all the square integrable martingales adapted to the sigma fields of $X_t$ is a more difficult problem. In fact, both Chou and Meyer [1] and Parthasarathy [5] have shown that, in general, there will be martingales that cannot be represented as stochastic integrals of $X_t$, at least not as stochastic integrals in the usual sense. In 1975 Gal'čuk [2] characterized all square integrable martingales adapted to the sigma fields of a process with stationary, independent increments, using the notion of stochastic integrals with respect to random measures. We give here a simple proof of this result. Our proof is completely elementary in that we use neither the Lévy–Khintchine formula, the Lévy kernel, the local characteristics, random measures, nor the 'uniqueness in law' of $X$. Instead, we use only Ito's lemma. In place of random measures we use the more conventional idea of stochastic integrals with an optional integrand.

In Section 2, we give the necessary preliminaries. In Section 3, we prove our main theorem, and in Section 4, we prove three extensions: an extension to the $d$-dimensional case, an improvement in the Poisson case, and an extension to the case of local martingales.

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## 2. Preliminaries

We recall briefly the necessary definitions and results. See [4] for details. Let $(\Omega, F, P)$ be a probability space, $F_t$ an increasing right continuous family of sub-sigma fields of $F$. Assume $F_0$ is trivial. A process $H$ defined on $\Omega\times[0,\infty)$ is called predictable if it is measurable with respect to the sigma field generated by the sets $\{(\omega, t)\colon S(\omega)<t\leq T(\omega), S, T$ stopping times$\}$. $H$ is optional if it is a measurable with respect to the sigma field generated by the sets $\{(\omega, t)\colon S(\omega)\leq t\leq T(\omega), S, T$ stopping times$\}$. $F_t$ is quasi-left continuous if $F_{T-}=F_T$ whenever $T$ is a stopping time such that $\{(\omega, T(\omega))\}$ is in the predictable sigma field.

Let $M_0=\{M_t\colon (M_t, F_t)$ is a martingale, $\sup_t \mathbf{E}M_t^2<\infty$, and $M_0=0\}$, the collection of square integrable martingales. There is a one-to-one correspondence between martingales in $M_0$ and $L_2$ random variables with mean 0 given by: if $M_t\in M_0$, $M_\infty=\lim_{t\to\infty}M_t$ is in $L_2(P)$, and $\mathbf{E}M_\infty=0$; if $Z$ is in $L_2(P)$ with $\mathbf{E}Z_0=0$, $M_t=\mathbf{E}(Z|F_t)$ is in $M_0$, and $M_\infty=Z$.

If $M\in M_0$, $M$ can be written as $M^c+M^d$, where $M^c$ has continuous paths, $M^d$ has paths of bounded variation, and $M^c, M^d\in M_0$. If $M\in M_0$, let $\langle M, M\rangle_t$ be the unique predictable process such that $M_t^2-\langle M, M\rangle_t$ is a martingale. Let $[M,M]_t=\langle M^c, M^c\rangle_t+\sum_{s\leq t}(\Delta M_s)^2$, where $\Delta M_s=M_s-M_{s-}$. If $M, N\in M_0$, let $\langle M, N\rangle=\frac{1}{4}(\langle M+N, M+N\rangle-\langle M-N, M-N\rangle)$, and similarly for $[M,N]$. If $X$ is a semimartingale, that is, if $X=M+A$, where $M\in M_0$ and $A$ is a process with paths of bounded variation, let $X^c=M^c$, and $[X,X]=\langle X^c, X^c\rangle+\sum_{s\leq t}(\Delta X_s)^2$. If $M\in M_0$, $H$ is predictable, and $\mathbf{E}\int_0^\infty H_s^2\,\mathrm{d}\langle M, M\rangle_s<\infty$, then $(H\cdot M)_t=\int_0^t H_s\,\mathrm{d}M_s$, the stochastic integral of $H$ with respect to $M$, is the unique martingale such that $\langle H\cdot M, N\rangle=H\cdot\langle M, N\rangle$ for all $N\in M_0$, where $(H\cdot\langle M,N\rangle)_t=\int_0^t H_s\,\mathrm{d}\langle M,N\rangle_s$, a Lebesgue–Stieltjes integral. If $H$ is optional, and $F_t$ is quasi-left continuous, the same definition holds provided $\langle,\rangle$ is replaced by $[,]$ throughout.

If the $F_t$ are quasi-left continuous, one can show that $\Delta(H\cdot M)_s=H_s\Delta M_s$. It follows that if $H$ is an optional process that is 0 except for countably many points, a.s., $H\cdot M$ is the unique martingale that has 0 continuous component and has paths of bounded variations with jumps $H_s\Delta M_s$.

If $X=M+A$ is a semimartingale, let $\int_0^t H_s\,\mathrm{d}X_s=\int_0^t H_s\,\mathrm{d}M_s+\int_0^t H_s\,\mathrm{d}A_s$, where the second integral is a Lebesgue–Stieltjes integral, provided, of course, that $H_s$ satisfies appropriate integrability conditions. We will need to use Ito's lemma:

**Lemma 2.1.** *If $X$ is a semimartingale, $f$ twice continuously differentiable, then*

$$f(X_t)-f(X_0)=\int_0^t f'(X_{s-})\,\mathrm{d}X_s+\tfrac{1}{2}\int_0^t f''(X_{s-})\,\mathrm{d}\langle X^c, X^c\rangle_s$$

$$+\sum_{s\leq t}[f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s],$$

*where $X^c$ is the continuous component of the martingale part of $X$.*

We will also use the integration by parts formula, an immediate corollary of Lemma 2.1:

**Corollary 2.2.** *If $X$ and $Y$ are semimartingales,*

$$X_tY_t = \int_0^t X_{s-}\,\mathrm{d}Y_s + \int_0^t Y_{s-}\,\mathrm{d}X_s + [X, Y]_t.$$

$M$ is a local martingale if $M_0 = 0$ and there exist stopping times $S_n \uparrow \infty$ such that $M_{t\wedge S_n}$ is a martingale and $\mathbf{E}|M_{S_n}| < \infty$ for each $n$. It can be shown that if $M$ is a local martingale, there exist stopping times $T_n \uparrow \infty$ such that $M_{t\wedge T_n} = U_t^n + V_t^n$, where $U^n \in M_0$ and $V^n$ is a martingale whose continuous component is 0 and which has a single jump of size $M_{T_n}$ at time $T_n$. One can then define $[M, M]$ by the same equation as before, and if the $F_t$ are quasi-left continuous and if $H$ is optional with $\mathbf{E}\int_0^\infty H_s^2\,\mathrm{d}[M, M]_s < \infty$, let $H\cdot M$ be the unique local martingale such that $[H\cdot M, N] = H\cdot[M, N]$ for all $N \in M_0$.

## 3. The basic theorem

We first prove a couple of lemmas.

**Lemma 3.1.** *Suppose $M, N \in M_0$, then*

(a) *if $L \in M_0$ and $\langle L, N\rangle = 0$ implies $\langle L, M\rangle = 0$, then $M = H\cdot N$ for some predictable process $H$.*

(b) *if the sigma fields $F_t$ are quasi-left continuous and $[L, N] = 0$ implies $[L, M] = 0$, then $M = H\cdot N$ for some optional process $H$.*

**Proof.** Let $H = \mathrm{d}\langle M, N\rangle/\mathrm{d}\langle N, N\rangle$, the Radon–Nikodym derivative, and let $L = M - H\cdot N$. Clearly $\langle L, N\rangle = 0$, and so $\langle L, M\rangle = 0$. Then $\langle L, L\rangle = \langle L, M\rangle - H\cdot\langle L, N\rangle = 0$, or $L = 0$.

The proof of (b) is the same with $\langle\,,\rangle$ replaced by $[\,,]$.

**Lemma 3.2.** *Suppose $M^n, M \in M_0$, $N$ a local martingale, $M_\infty^n \to M_\infty$ in $L_2$, and $M^n = H^n\cdot N$, where $H^n$ is predictable and satisfies $\mathbf{E}\int_0^\infty (H_s^n)^2\,\mathrm{d}[N, N]_s < \infty$. Then $M = H\cdot N$ for some $H$ predictable such that $\mathbf{E}\int_0^\infty H_s^2\,\mathrm{d}[N, N]_s < \infty$.*

*If the sigma fields $F_t$ are quasi-left continuous, the result still holds if 'predictable' is replaced by 'optional'.*

**Proof.** $\mathbf{E}(M_\infty^n - M_\infty^m)^2 = \mathbf{E}[M^n - M^m, M^n - M^m]_\infty = \mathbf{E}\int_0^\infty (H_s^n - H_s^m)^2\,\mathrm{d}[N, N]_s$. By the completeness of the space $L_2$ with respect to the measure on $R\times\Omega$ given by $\mathbf{E}\int_0^\infty \cdot\,\mathrm{d}[N, N]_s$, there is a predictable (optional) process $H$ such that

$$\mathbf{E}[H\cdot N - M^n, H\cdot N - M^n]_\infty = \mathbf{E}\int_0^\infty (H_s - H_s^n)^2\,\mathrm{d}[N, N]_s \to 0.$$

Hence we have $\mathbf{E}[(H\cdot N)_\infty - M_\infty]^2 = 0$, or $M_\infty = (H\cdot N)_\infty$, which gives our result.

Suppose $X_t$ is a process with stationary, independent increments with paths that are right continuous with left limits, $X_0 = 0$, $F_t$ the sigma fields generated by $X_t$. Then $F_0$ is trivial, $F_t$ is quasi-left continuous, $X_t$ is an optional process, and the process $X_{t-}$ is predictable. If $\mathbf{E}|X_t| < \infty$, let $X'_t = X_t$. If $\mathbf{E}|X_t| = \infty$, let

$$X'_t = X_t - \sum_{s\le t} \Delta X_s 1_{(|\Delta X_s|\ge 1)} + \sum_{s\le t} f(\Delta X_s),$$

where $f(x) = 1 + (2/\pi)\arctan x$ if $x \ge 1$, $-1-(2/\pi)\arctan|x|$ if $x \le -1$, and 0 otherwise. It is clear that $X'_t$ has stationary, independent increments and that $X'_t$ generates the same sigma field as $X_t$. In the case $\mathbf{E}|X_t| = \infty$, $X'_t$ has bounded jumps, and so $\mathbf{E}|X'_t| < \infty$. Let $X''_t = X'_t - \mathbf{E}X'_t$. Since $\mathbf{E}X'_t$ is constant for each $t$, $X''_t$ and $X'_t$ generate the same sigma fields. Hence $X''_t$ is a process with stationary, independent increments that generates the same sigma fields as $X_t$. Note that $X''_t = X_t$ in the case $\mathbf{E}|X_t| < \infty$ and $\mathbf{E}X_t = 0$. Note also that $X''_t$ is a local martingale. We may therefore assume without loss of generality that $\mathbf{E}|X_t| < \infty$ and $\mathbf{E}X_t = 0$.

**Theorem 3.3.** *Suppose $X_t$ is a process with stationary, independent increments, $\mathbf{E}|X_t| < \infty$ and $\mathbf{E}X_t = 0$. Then $M \in M_0$ if and only if there exists an optional process $H$ such that $M = H\cdot X$ and $\mathbf{E}\int_0^\infty H_s^2\, \mathrm{d}[X,X]_s < \infty$. $H$ is given by $\mathrm{d}[M,X]/\mathrm{d}[X,X]$ and is unique up to a.s. equivalence with respect to the measure on $\Omega\times[0,\infty)$ given by $\mathbf{E}\int_0^\infty \cdot\, \mathrm{d}[X,X]_s$.*

**Proof.** The 'if' part follows from the definition of stochastic integral. If $M = H\cdot X$, $[M,X] = H\cdot[X,X]$, which gives the formula for $H$. If $M = H\cdot X = H'\cdot X$, then

$$\begin{aligned} 0 &= \mathbf{E}((H\cdot X)_\infty - (H'\cdot X)_\infty)^2 = \mathbf{E}[(H-H')\cdot X, (H-H')\cdot X]_\infty \\ &= \mathbf{E}((H-H')^2\cdot[X,X])_\infty, \end{aligned}$$

which gives uniqueness.

For the 'only if' part, suppose first that $X$ has bounded jumps. Let $N$ be a positive integer, and let $Y_t = X_{t\wedge N}$, and let $L \in M_0$ such that $[L, Y] = 0$ for all $t$. We first prove by induction on $r$ that if $u$ is any positive real $\le N$ and $Z_t = \mathbf{E}(Y_u^r | F_t)$, then $[M, Z]_t = 0$ for all $t$. $r = 1$ is clear.

If $r > 1$, by Lemma 2.1,

$$Y_u^r = \int_0^u rY_{s-}^{r-1}\,\mathrm{d}Y_s + \tfrac{1}{2}r(r-1)\int_0^u Y_{s-}^{r-2}\,\mathrm{d}\langle Y^c, Y^c\rangle_s$$

$$+ \sum_{s\le u} [Y_s^r - Y_{s-}^r - rY_{s-}^{r-1}\,\Delta Y_s].$$

(i) $[L, rY_-^{r-1}\cdot Y] = rY_-^{r-1}\cdot[L, Y] = 0$.

(ii) Since the continuous part of $Y$, $Y^c$, is Brownian motion up to time $N$, $\langle Y^c, Y^c\rangle_t = at$, where $a = \mathbf{E}(Y_1^c)^2$, if $t \le N$. Then by the induction hypotheses and the fact that $Y_s^{r-2}$ is right continuous,

$$\left[L, \int_0^u Y_{s-}^{r-2}\,\mathrm{d}s\right] = \left[L, \int_0^u Y_s^{r-2}\,\mathrm{d}s\right] = \int_0^u [L, Y_s^{r-2}]\,\mathrm{d}s = 0.$$

(iii) Since $Y_s = Y_{s-} + \Delta Y_s$, by the binomial theorem $Y_s^r - Y_{s-}^r - rY_{s-}^{r-1}\,\Delta Y_s$ is a polynomial in $Y_{s-}$ and $\Delta Y_s$. So it suffices to show that $[L, Z] = 0$, where $Z = \mathbf{E}(\sum_{s\leq u} Y_{s-}^p\,\Delta Y_s^q \mid F_t)$ and $p \leq r-2$, $q \geq 2$. If $W_t = \sum_{s\leq t}\Delta Y_s^q$, $W$ has stationary, independent increments, and $W_t - bt$ is a martingale, where $b = \mathbf{E}W_1$. Then $W_t - bt = \int_0^t \Delta Y_s^{q-1}\,\mathrm{d}Y_s$, since both sides are martingales with paths of bounded variation with the same size jumps.

$$Z_u = \sum_{s\leq u} Y_{s-}^p\,\Delta Y_s^q = \int_0^u Y_{s-}^p\,\mathrm{d}W_s = \int_0^u Y_{s-}^p\,\mathrm{d}(W_s - bs) + b\int_0^u Y_{s-}^p\,\mathrm{d}s$$
$$= \int_0^u Y_s^p\,\Delta Y_s^{q-1}\,\mathrm{d}Y_s + b\int_0^u Y_s^p\,\mathrm{d}s.$$

Since $p \leq r-2$, using an argument similar to (i) for the first term and an argument similar to (ii) for the second term shows that $[Z, L] = 0$.

Next, by Lemma 3.1 for each $r$ and $u$, $\mathbf{E}(Y_u^r \mid F_t) = \mathbf{E}Y_u^r + H^{r,u}\cdot Y$, for some optional process $H^{r,u}$. Since $Y_t$ has bounded jumps,

$$\mathbf{E}(\exp(ikY_u) \mid F_t) = \mathbf{E}\exp(ikY_u) + H^u\cdot Y$$

for some optional process $H^u$ by Lemma 3.2. Since $Y_{N-} = Y_N$, a.s., we may assume $H_t^u$ is 0 if $t \geq N$.

If $s$ is a fixed real, repeating the argument for $X_t' = X_{t+s} - X_s$, $F_t' = \sigma(X_u'; u \leq t)$ shows that $\exp(ik'(X_{t+s} - X_s))$ is also of the form: constant $+ H'\cdot X$, where $H'$ is 0 on $[0, s)$ and $(t+s, \infty)$.

If $N = H\cdot X$ and $N = H'\cdot X$, where $H$ is 0 on $[a, b]$ and $H'$ is 0 on $[a, b]^c$, then $1_{[a,b]}\cdot N = N$, $1_{[a,b]^c}\cdot N' = N'$, and by Corollary 2.2,

$$NN' = N_-\cdot N' - N'_-\cdot N + [N, N']$$
$$= N_-H'\cdot X + N'_-H\cdot X + 1_{[a,b]}1_{[a,b]^c}\cdot[N, N'] = (N_-H' + N'_-H)\cdot X,$$

again a stochastic integral of $X$. It follows easily that if $s_0 \leq s_1 \leq \cdots \leq s_n$, then for reals $k_1, \ldots, k_n$, $\prod_{j=1}^n \exp(ik_j(X_{s_j} - X_{s_{j-1}}))$ is a stochastic integral of $X$.

Finally, let $J_u = \sum_{s\leq u}\Delta X_s 1_{(|\Delta X_s|\geq N)}$, and let $X_u^N = X_u - J_u + \mathbf{E}J_u$. $X_u^N$ has independent increments, mean 0, and bounded jumps. Hence

$$Z = \prod_{j=1}^n \exp(ik_j(X_{s_j}^N - X_{s_{j-1}}^N)) = \mathbf{E}Z + H\cdot X^N \quad \text{for some } H.$$

Let $B = \{(t, \omega): |\Delta X_t(\omega)| \geq N\}$. It is clear that $B$ is an optional set, and $X^N = 1_{B^c}\cdot X$. Hence $Z = \mathbf{E}Z + H1_{B^c}\cdot X$, a stochastic integral of $X$. Letting $N\to\infty$ shows by Lemma 3.2 that $\prod_{j=1}^n \exp(ik_j(X_{s_j} - X_{s_{j-1}}))$ is a stochastic integral of $X$, and using Lemma 3.2 again, any square integrable martingale adapted to $F_t$ is, also.

## 4. Extensions

(i) Suppose $X_t$ is $d$-dimensional.

**Lemma 4.1.** *Suppose the sigma fields $F_t$ are quasi-left continuous. Suppose $N^1, \ldots, N^d \in M_0$, $M^n \in M_0$ for each $n$, $M^n = \sum_{j=1}^d H_{jn} \cdot N^j$ for some optional processes $H_{jn}$, and $M_\infty^n \to M_\infty$ in $L_2$. Then $M = \sum_{j=1}^d H_j \cdot N^j$ for some optional processes $H_j$.*

**Proof.** First orthogonalize the $N$'s by a Gram–Schmidt process. Let $L^1 = N^1$, and for $2 \leq j \leq d$, let

$$L^j = N^j - \sum_{k=1}^{j-1} \frac{\mathrm{d}[N^j, L^k]}{\mathrm{d}[L^k, L^k]} \cdot L^k.$$

Let

$$J = M - \sum_{j=1}^{d} \frac{\mathrm{d}[M, L^j]}{\mathrm{d}[L^j, L^j]} \cdot L^j.$$

It should be clear that it suffices to show $J = 0$. $[J, L^j] = 0$ for all $j$, hence $[J, N^j] = 0$ for all $j$, and so $[J, M^n] = 0$ for all $n$.

Then

$$[J, J] = [J, M] = [J, M - M^n] + [J, M^n] \leq [J, J]^{1/2}[M - M^n, M - M^n]^{1/2},$$

using the inequality of Kunita and Watanabe. Since $\mathbf{E}[M - M^n, M - M^n] \to 0$, we have our result.

**Theorem 4.2.** *Suppose $X_t = (X_t^1, \ldots, X_t^d)$ is a $d$-dimensional process with stationary independent increments, $\mathbf{E}(X_t^j)^2 < \infty$ for all $j$, and $\mathbf{E}X_t = 0$. If $M$ is adapted to the sigma fields of $X_t$ and $M \in M_0$, then there exist optional processes $H_1, \ldots, H_d$ such that $M = \sum_{j=1}^d H_j \cdot X^j$.*

*Comment.* If $\mathbf{E}(X_t^j)^2 = \infty$ for some $j$, or $\mathbf{E}X_t \neq 0$, replace $X_t$ by $X_t''$ in a way similar to that preceding Theorem 3.3.

**Proof.** Let (,) denote the usual inner product. If $u = (u_1, \ldots, u_d)$ is any vector, $(u, X_t)$ is a process with stationary independent increments, and by Theorem 2.3

$$\exp(i(u, X_t)) - \mathbf{E}\exp(i(u, X_t)) = \int_0^t H_s \,\mathrm{d}((u, X_s)) = \sum_{j=1}^{d} \int_0^t u_j H_s \,\mathrm{d}X_s^j$$

for some optional process $H$. As in the proof of Theorem 2.3, if $M = \sum_{j=1}^d H_j \cdot X_j$ and

$N = \sum_{j=1}^{d} K_j \cdot X^j$, where $1_{[a,b]} \cdot H_j = H_j$ and $1_{[a,b]^c} \cdot K_j = K_j$ for all $j$, then

$$\begin{aligned} MN &= \sum_{j,k} (H_j \cdot X^j)(K_k \cdot X^k) \\ &= \sum_{j,k} ((H_j \cdot X^j)_- K_k \cdot X^k + (K_k \cdot X^k)_- H_j \cdot X^j \\ &\quad + 1_{[a,b]} 1_{[a,b]^c} \cdot [H_j \cdot X^j, K_k \cdot X^k]). \end{aligned}$$

Since the final term in each summand is 0, $MN$ is also of the form $\sum_{j=1}^{d} I_j \cdot X^j$ for some optional processes $I_j$.

Hence, if $s_0 \leqslant s_1 \leqslant \cdots \leqslant s_k$ are positive reals, $u_1, \ldots, u_k$ are vectors, and $Z = \prod_{m=1}^{k} \exp(i(u_m, X_{s_m} - X_{s_{m-1}}))$, then $Z - \mathbf{E}Z$ is of the form $\sum_{j=1}^{d} H_j \cdot X^j$. But the set of such random variables $Z - \mathbf{E}Z$ is dense in $M_0$ in $L_2$-norm, and the result now follows easily from Lemma 4.1.

(ii) Chou and Meyer [1] and Parthasarathy [5] have both pointed out that one cannot, in general, take $H$ to be predictable. If, however, $X_t = P_t - (\mathbf{E}P_1)t$, where $P_t$ is a Poisson process, one can always take $H$ predictable. Here $X_t^n \equiv X_t$. Following the proof of Theorem 3.3, it suffices to show that $\mathbf{E}(X_N^r \mid F_t) = \mathbf{E}X_N^r + \int_0^t H_s \, \mathrm{d}X_s$ for some $H$ predictable. Choosing an $L$ such that $\langle L, X \rangle = 0$ (instead of $[L, X] = 0$), it suffices to show that $\langle L, \Delta X^{q-1} \cdot X \rangle = 0$. Hence if we show $\Delta X^{q-1} \cdot X = K \cdot X$ for some predictable $K$, any integer $q \geqslant 1$, we would have $\langle L, K \cdot X \rangle = K \cdot \langle L, X \rangle$, and we would be done. Since all the jumps of $X$ are the same size, $c$, $\Delta X^{q-1} = c^{q-2} \Delta X$, and

$$(\Delta X \cdot X)_t = \sum_{s \leqslant t} \Delta X_s^2 - kt = \sum_{s \leqslant t} c \, \Delta X_s - kt = cX_t = (c \cdot X)_t,$$

where $k = \mathbf{E}(\sum_{s \leqslant 1} \Delta X_s^2) = \mathrm{Var}(X_1) = c\,\mathbf{E}(\sum_{s \leqslant 1} \Delta X_s)$.

(iii) Suppose $M$ is a local martingale adapted to the sigma fields of $X_t$ with $M_0 = 0$. Again, we may assume $\mathbf{E}|X_t| < \infty$, $\mathbf{E}X_t = 0$. Since $M$ is a local martingale, there exist stopping times $T_n$ such that $M_{t \wedge T_n} = U_t^n + V_t^n$, where $U^n \in M_0$ and $V^n$ is a martingale whose continuous component is 0 which has a single jump of size $M_{T_n}$ at time $T_n$.

Since the sigma fields of $F_t$ are quasi-left continuous, any stochastic integral of $X$ jumps only when $X$ does, hence any martingale in $M_0$ jumps only when $X$ does, from which it follows easily that $V^n$ jumps only when $X$ does. If $U^n = H \cdot X$, then $M_{T_n} = (H + (M_{T_n}/\Delta X_{T_n}) 1_{(\Delta X_{T_n} \neq 0)}) \cdot X$.

It is now clear that every local martingale $M$ is a stochastic integral with respect to $X$.

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